Exam Calculus 1

November 4, 2017: 10.00-13.00.

This exam has 8 problems. Each problem is worth 1 point. Total: $8 + 1$ (free) $= 9$ points; more details can be found below. Write on each page your name and student number. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by work. Success.

1. (a) Formulate the principle of mathematical induction.

   (b) Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$$

   for every natural number $n > 1$.

2. Find all complex solutions z satisfying

$$\left(\frac{1+i}{z}\right)^3 = 8i$$

3. (a) The function f is defined on some open interval that contains the number a, except possibly at a itself. Give the precise definition of

$$\lim_{x \to a} f(x) = L$$

   (b) Prove, using this definition, that

$$\lim_{x \to 1} 7x + 2 = 9$$

4. The Generalized Mean Value Theorem states that if the functions f and g are both continuous on $[a, b]$ and differentiable on (a, b), and if $g'(x) \neq 0$ for every x in (a, b), then there exists a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

   (a) Formulate the Mean Value Theorem.

   (b) Proof the Generalized Mean-Value Theorem by applying the Mean Value Theorem to

$$h(x) = (f(b) - f(a))\,(g(x) - g(a)) - (g(b) - g(a))\,(f(x) - f(a))$$

(c) What is wrong with the following "proof" of the Generalized Mean Value Theorem? By the Mean Value Theorem, $f(b) - f(a) = (b-a)f'(c)$ for some c between a and b, and similarly $g(b) - g(a) = (b-a)g'(c)$ for some such c. Hence, $(f(b) - f(a))/(g(b) - g(a)) = f'(c)/g'(c)$, as required.

5. Evaluate

(a)

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

(b)

$$\lim_{x \to 0+} x^{\sqrt{x}}$$

6. (a) Show that the values of the following expression do not depend on x:

$$\int_0^x \frac{1}{1+t^2}\,dx + \int_0^{1/x} \frac{1}{1+t^2}\,dx$$

(b) Find all continuous functions $f(x)$ satisfying

$$\int_0^x f(t)\,dt = (f(x))^2 + c$$

for some constant c

7. Evaluate

(a)

$$\int e^x \sin x\,dx$$

(b)

$$\int_0^1 \sqrt{x}\,\ln x\,dx$$

8. Solve the initial value problem

$$y'(x) + \frac{y}{x} = 2 \qquad y(1) = 0$$

Maximum points:

1a	0.4	2	1.0	3a	0.4	4a	0.2	5a	0.5	6a	0.5	7a	0.5	8	1.0
b	0.6			b	0.6	b	0.6	b	0.5	b	0.5	b	0.5		
						c	0.2								